(Neutral) Functional Differential Equations with (Infinite) Delay

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$$
\begin{cases}
\dfrac{\partial}{\partial t}\left[v(t,\xi) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta,\xi))d\theta \right] \\
= \\
\dfrac{\partial^2}{\partial \xi^2}\left[v(t,\xi) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta,\xi))d\theta \right] \\
\quad + \int_{-\infty}^{0} K_2(\theta, v(t+\theta,\xi))d\theta, \quad t \geq 0,\, 0 \leq \xi \leq 1, \\
v(t,0) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta,0))d\theta = \\
v(t,1) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta,1))d\theta = 0,\ \ t \geq 0, \\
v(\theta,\xi) = v_0(\theta,\xi),\ \ -\infty < \theta \leq 0,\ \ 0 \leq \xi \leq 1,
\end{cases}
$$

A model of transmission lines on the unit circle

J. Wu, Theory and Applications of Partial FDEs, Springer-Verlag, (1996).

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$$\frac{\partial}{\partial t}[x(.,t) - qx(.,t-r)]$$

=

$$k\frac{\partial^2}{\partial \xi^2}[x(.,t) - qx(.,t-r)] + f(x_t),$$

$k, r > 0$ and $0 \leq q < 1$,
$x_t(\xi, \theta) = x(\xi, t+\theta)$ for
$-r \leq \theta \leq 0,\ t \geq 0,\ \xi \in S^1$

$$\frac{\partial}{\partial t}\mathrm{D}v_t = k\frac{\partial^2}{\partial x^2}\mathrm{D}v_t + f(v_t), \quad t \geq 0,$$

$\mathrm{C} = \mathrm{C}([-r, 0]; H^1(S^1))$ as phase space

$\mathrm{D} \in \mathrm{L}(\mathrm{C}, H^1(S^1))$; $\mathrm{D}\varphi = \varphi(0) - P\varphi$.

$A = k\frac{\partial^2}{\partial x^2}$ with domain $H^2(S^1)$

yields an i. g. of a C_0-semigroup on $H^1(S^1)$.

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Analytic Semigroup

$$\mathrm{Im}(G) \subseteq D((-A)^{\beta}),\ 0 < \beta < 1$$

$$\begin{cases} \dfrac{d}{dt}(x(t) - G(t, x_t)) \\ = & t \geq 0, \\ A(x(t) - G(t, x_t)) + F(t, x_t), \\ x_0 = \varphi, \end{cases}$$

A is a Hille-Yoshida operator on E,

C $=$ C$([-r, 0]; E)$ as phase space,

$G, F : [0, +\infty) \times$ C $\to E$ continuous

$\forall\, t \geq 0,\ x_t \in$ C is defined by

$$x_t(\theta) = x(t + \theta), \text{ for } \theta \in [-r, 0].$$

$$\mathsf{L}_\gamma = \left\{ \begin{array}{c} \phi : (-\infty, 0] \to E \text{ meas.} \\ \text{s. t. } e^{-\gamma \cdot}\phi(.) \text{ is integrable} \end{array} \right\}$$

with the seminorm

$$\|\phi\|_\gamma := |\phi(0)| + \int_{-\infty}^{0} e^{-\gamma\theta}|\phi(\theta)|.$$

$$C_\gamma = \left\{ \begin{array}{c} \phi : (-\infty, 0] \to E \text{ such that} \\ \lim\limits_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E \end{array} \right\}$$

w/ the norm $\|\phi\|_\gamma := \sup\limits_{\theta \le 0} e^{\gamma\theta}|\phi(\theta)|,$

Axioms (A), (A1) and (B).

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Phase space for retarded equations with infinite delay,
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Hypothesis: Phase Space

$(B, \|\cdot\|_B)$ is an axiomatic abstract (semi)normed linear space mapping $(-\infty,0]$ into E.

→ (**H1**) A is Hille-Yosida, i.e.,

$\exists\ \bar{M} \geq 1,\ \bar{\omega} \in$ R such that

$(i)\ (\bar{\omega}, +\infty) \subset \rho(A),$

$(ii)\ \forall\ \lambda > \bar{\omega},\ n \in$ N

$$(\lambda - \bar{\omega})^n \| (R(\lambda, A))^n \| \leq \bar{M}.$$

$\rho(A)$ is the resolvent set of A

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

Definition 1. $\forall\ \varphi \in \mathrm{B}$, we say that $x : (-\infty, a] \to E,\ a > 0,$ is an integral solution in $(-\infty, a]$ if:

(i) x is continuous on $[0, a]$,

(ii) $x(t) = \varphi(t), \quad -\infty < t \le 0,$

(iii) for all $t \in [0, a]$,

$\int_0^t (x(s) - G(s, x_s))ds \in D(A)$, and

$$x(t) = G(t, x_t) + \varphi(0) - G(0, \varphi) + A\int_0^t (x(s) - G(s, x_s))ds + \int_0^t F(s, x_s)ds.$$

(H2) $G : [0, +\infty) \times \mathrm{B} \to E$ is cont. and $\exists$ $\alpha_0 > 0$ such that $\alpha_0 K(0) < 1$ and

$$|G(t,\varphi_1) - G(t,\varphi_2)| \leq \alpha_0 \|\varphi_1 - \varphi_2\|_{\mathrm{B}}.$$

(H3) $F : [0, +\infty) \times \mathrm{B} \to E$ is cont. and $\exists$ $\beta_0 > 0$ such that

$$|F(t,\varphi_1) - F(t,\varphi_2)| \leq \beta_0 \|\varphi_1 - \varphi_2\|_{\mathrm{B}}.$$

(H1)---(H3) with φ(0)-G(0,φ) $\in \overline{D(A)}$
give global existence and uniqueness.

$$\begin{cases} \frac{\partial}{\partial t} \mathcal{D}u_t = A\mathcal{D}u_t + F(t, u_t), \ t \geq 0 \\ u_0 = \phi \in \mathcal{B}, \end{cases}$$

$$D\varphi = \varphi(0) - D_0\varphi \text{ for any } \varphi \in B,$$

$D_0 \in L(B, E)$

$\forall u : (-\infty, b] \to E$, $b > 0$, & $t \in [0, b]$,

$u_t : (-\infty, 0] \to E$ defined by

$$u_t(\theta) = u(t + \theta) \text{ for } \theta \in (-\infty, 0].$$

Let X := { $\varphi \in \mathcal{B} : D\varphi \in \overline{D(A)}$ }

What is a Global Attractor ?

X is a Banach Space

Suppose (U(t)) is a semigroup of operators w/

- (U(t)), $t \geq 0$, maps bounded sets of X into bounded sets of X

Under (U(t)), $t \geq 0$,

A global attractor A in X is a maximal compact invariant set that attracts each bounded set B in X, that is

$$\text{dist}(U(t)\,B\,,\,A\,) \to 0 \text{ as } t \to +\infty$$

Global Attractor: Conditions ?

1. A semigroup $(U(t)),\ t \geq 0$, on **X** is **asymtotically smooth** if for any nonempty closed bounded set **B** in **X** such that $U(t)B \subset B$ there exists a compact set C in **B** such that C attracts **B**, that is,

$$\mathrm{dist}(U(t)B, C) \to 0 \text{ as } t \to +\infty$$

$\Uparrow$

$(U(t)),\ t \geq 0$, is **quasi-compact**

Hence

Let **X** be the space of initial functions ϕ, that is, **X** $:= \{ \varphi \in \mathcal{B} : D\varphi \in \overline{D(A)} \}$

Suppose

1. F depends only on ϕ

2. F satisfies enough conditions to ensure that it maps bounded sets of **X** into bounded sets of E and for each initial function ϕ in **X**, the eq. has a unique global integral solution and this solution is continuous in ϕ.

Solution Semigroup

Set $U(t)\,\phi = u_t(., \phi)$, $t \geq 0$,

where $u(., \phi)$ is the unique integral solution issued from ϕ in **X** with

$$X := \{\phi \in \mathcal{B} : D\phi \text{ in the closure of } D(A)\}$$

(U(t)) is a semigroup of operators

Suppose

* (U(t)), $t \geq 0$, maps bounded sets of **X** into bounded sets of **X**

(U(t)) is an α-contraction:

U(t) = V(t) + W(t), t > 0

V(t), t>0, is compact

(W(t)), t ≥ 0, is exp.tially stable

⇑?

(H4) $T_0(t)$, t>0, is compact and exponentially stable (α)

(H5)D is stable (not suffi.ct even w/ M, K bounded on [0,+∞))

(H2') D is $\mathcal{B}$-unif. exp. stable

Again

Under $(U(t)), t \geq 0$,

A **global attractor** $\mathcal{A}$ in X is a maximal compact invariant set that attracts each bounded set B in X, that is

$$\mathrm{dist}(U(t)\,B\,,\,\mathcal{A}\,) \to 0 \text{ as } t \to +\infty$$

Standard Case: D $\phi = \phi(0)$ (Not Neutral) with $B = C_\gamma$

$(U(t))$ is an **α-contraction**:

$U(t) = V(t) + W(t)$, $t > 0$ and

$V(t)$, $t>0$, is compact

$(W(t))$, $t \geq 0$, is exp. stable

$\Uparrow$

(H4) $T_0(t)$, $t>0$, is compact & exp. stable

zero is exp. stable

$\Uparrow$

F is differentiable at 0_B and $F'(0_B)=0_E$

and

$s_1(A_T)<0$

Hypotheses:

(U(t)), $t \geq 0$, is point dissipative

(compact dissipative) if:

there is a bounded set B in X that attracts each point of X (each compact of X),

That is,

for all ϕ in X (C cpct in X)

$\text{dist}(U(t)\phi \, , B) \to 0$ as $t \to +\infty$

$\left(\text{dist}(U(t)C \, , B) \to 0 \text{ as } t \to +\infty\right)$

4 Application

To illustrate Theorem 1.2 that we have developed, we consider the following reaction diffusion equation with infinite delay

$$\begin{cases} \frac{\partial}{\partial t} w(t,\xi) = \frac{\partial^2}{\partial \xi^2} w(t,\xi) - \mu w(t,\xi) + \int_{-\infty}^{0} k(\theta) g(w(t+\theta,\cdot))(\xi) d\theta, \ t \geq 0, \ 0 \leq \xi \leq \pi, \\ w(t,0) = w(t,\pi) = 0, \ t \geq 0, \\ w(\theta,\xi) = w_0(\theta,\xi), \ -\infty < \theta \leq 0, \ 0 \leq \xi \leq \pi, \end{cases} \tag{4.1}$$

where μ is a positive constant, $w_0(t,\cdot) \in E := C([0,\pi];\mathbb{R})$ a Banach space with the supreme norm, $g : E \to E$ is globally Lipschitz continuous with Lipschitzian constant $l > 0$ and $k : (-\infty, 0] \to [0, +\infty)$ satisfies

$$l \int_{-\infty}^{0} e^{-\gamma\theta} k(\theta) d\theta < \mu.$$

For some $\gamma > \mu$, define

$$C_\gamma = \left\{ \phi \mid \phi : (-\infty, 0] \to E \text{ is continuous and } \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists} \right\}$$

endowed with norm $\|\phi\|_\gamma = \sup_{-\infty < \theta \leq 0} e^{\gamma\theta}\|\phi(\theta)\|$, $\phi \in C_\gamma$. Define

$$\begin{cases} Ay = y'' - \mu y, \quad y \in D(A), \\ D(A) = \{y \in C^2([0,\pi] : \mathbb{R}) : y(0) = y(\pi) = 0\}, \end{cases}$$

and $F : C_\gamma \to E$ with

$$F(\phi)(\xi) = \int_{-\infty}^{0} k(\theta) g(\phi(\theta))(\xi) d\theta, \quad \phi \in C_\gamma, \quad \xi \in [0,\pi].$$

We can see that

$$\overline{D(A)} = \{y \in E : y(0) = y(\pi) = 0\} \neq E.$$

Set $x(t) = w(t, \cdot) \in E$, $x_t(\theta) = x(t+\theta)$, $x_t \in C_\gamma$ and $\phi(\theta)(\xi) = w_0(\theta, \xi)$. Then Eq.(4.1) can be rewritten into the following abstract Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + F(x_t), \quad t \geq 0, \\ x_0 = \varphi \in C_\gamma. \end{cases} \tag{4.2}$$

(H3) $T_0(t)$, defined by (1.3), is compact for $t > 0$

(H4)$'$ $\|T_0(t)\|_{\mathcal{L}} \leq e^{-\alpha t}$, $t \geq 0$, with a constant $\alpha > 0$.

Theorem 1.2 *Under the assumptions (H1)-(H3) and (H4)$'$ with* $\min\{\gamma, \alpha\} > L$, *Eq.(1.1) has a global attractor* $\mathcal{A}$.

Now let us verify the assumptions in Theorem 1.2 for Eq.(4.2). In [31], the authors have proved that, for the operator $By = y''$ with $D(B) = D(A)$,

$$(0, +\infty) \subset \rho(B) \quad \text{and} \quad \|(\lambda I - B)^{-1}\|_{\mathcal{L}} \leq \frac{1}{\lambda} \quad \text{for} \quad \lambda > 0.$$

Therefore, we have

$$(-\mu, +\infty) \subset \rho(A) \quad \text{and} \quad \|(\lambda I - A)^{-1}\|_{\mathcal{L}} \leq \frac{1}{\lambda + \mu} \quad \text{for} \quad \lambda > -\mu, \tag{4.3}$$

which implies that A is a Hille-Yosida operator.

Furthermore, we denote B_0 as the part of B on $\overline{D(B)}$, i.e.,

$$B_0(y) = B(y), \quad \text{for} \quad y \in D(B_0) = \{y \in D(B) : y''(0) = y''(\pi) = 0\}.$$

Then B_0 is a densely defined Hille-Yosida operator and generates a compact C_0-semigroup $T_{B_0}(t)$, $t \geq 0$, on $\overline{D(B)}$ with $\|T_{B_0}(t)\|_{\mathcal{L}} \leq 1$, see [16]. Consequently, A_0, the part of A on $\overline{D(A)}$, generates a compact C_0-semigroup $T_0(t)$, $t \geq 0$, such that

$$\|T_0(t)\|_{\mathcal{L}} \leq e^{-\mu t}. \tag{4.4}$$

On the other hand, for every $\phi_1, \phi_2 \in C_\gamma$, we have

$$\begin{aligned}
\|F(\phi_1) - F(\phi_2)\| &= \sup_{0\le\xi\le\pi} |F(\phi_1)(\xi) - F(\phi_2)(\xi)| \\
&\le \sup_{0\le\xi\le\pi} \left| \int_{-\infty}^{0} k(\theta)(g(\phi_1(\theta))(\xi) - g(\phi_2(\theta))(\xi))d\theta \right| \\
&\le \sup_{0\le\xi\le\pi} \int_{-\infty}^{0} k(\theta) l \left| \phi_1(\theta)(\xi) - \phi_2(\theta)(\xi) \right| d\theta \\
&\le l \int_{-\infty}^{0} e^{-\gamma\theta} k(\theta) d\theta \|\phi_1 - \phi_2\|_\gamma,
\end{aligned}$$

i.e., F is globally Lipschitz continuous with Lipschitzian constant

$$L := l \int_{-\infty}^{0} e^{-\gamma\theta} k(\theta) d\theta < \mu. \tag{4.5}$$

Since $T_0(t)$ is compact for $t > 0$, based on (4.3), (4.4) and (4.5) and applying Theorem 1.2, we obtain that Eq.(4.1) has a global attractor.

Theorem 1.2 *Under the assumptions (H1)-(H3) and (H4)′ with* $\min\{\gamma, \alpha\} > L$, *Eq.(1.1) has a global attractor* $\mathcal{A}$.

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